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Unified kinematics of bradyons and tachyons in six-dimensional space–time

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Abstract. An explicit form of the transformation matrix in six-dimensional space–time M_6 is given both for subluminal and superluminal cases. Contraction into four dimensions provides the usual equations. Though a superluminal transformation in six dimensions is either real or complex, it must be necessarily complex when contracted into four dimensions if one considers the Minkowski subspace M_4 as being the same for a subluminal and for a superluminal observer. The coordinates which become imaginary under the transformation reflect the fact that the events observable to one observer are not observable to the other. If there were only four dimensions in the world this would be difficult to understand. On the other hand, in six dimensions we can say that M_4 is a subspace of events which are observable to a given observer. Another observer, moving faster than light, then doesn't observe the same subspace of events, but some other subspace M'_4. The cross section $M'_4 \cap M_4$ is observable to both observers. In the second part of the paper it is shown how we can use a single formalism for both bradyons and tachyons.

1. Introduction

Recently, we have been faced with an increasing interest in the investigation of special relativity in three-dimensional space and three-dimensional time (Dorling 1970, Demers 1975, Kalitzin 1975, Mignani and Recami 1976, Pappas 1978, Cole 1977, 1978, 1980). The introduction of three-dimensional time is appealing, since it restores the symmetry between space and time. Such a symmetry is especially useful when studying possible extensions of the Lorentz transformations to frames and objects moving faster than light (Recami and Mignani 1974 and references therein). Until recently the issues were controversial. The situation has been clarified by Cole (1980), who correctly understood the essence of six-dimensional transformations and their contraction into the usual four-dimensional transformations. Here I develop and systematise his work further, and write down an explicit form of the transformation matrix. I start from certain general principles concerning the distinction between subluminal and superluminal objects (bradyons and tachyons) and I show that both bradyons and tachyons can be treated on the unified footing. The essence is the introduction of a new degree of freedom, *scale*, by which a tachyon distinguishes itself from a bradyon. The transformation which maps a bradyon at rest to a tachyon with infinite speed is a suitable dilatation. A general superluminal transformation is the product of the dilatation and a subluminal transformation with properly chosen parameters. Within the proposed formalism we do not need to write separate equations for bradyons and tachyons. Both types of particles and corresponding transformations obey the same unified equations.

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2. Three-dimensional space and three-dimensional time

The necessity of time being more than one dimensional has been discussed elsewhere (Pavšič 1981). Here I shall just assume that events form a six-dimensional pseudo-Euclidean non-compact continuum M_6 which is the direct sum

$$M_6 = T_3 \oplus E_3 \tag{2.1}$$

of three-dimensional Euclidean time T_3 and three-dimensional Euclidean space E_3. Let the metric of M_6 have the signature $(+++---)$.

I shall also assume that by our immediate perception, *in the absence of tachyons*, we are not aware of the three dimensionality of time. An observer perceives the time as being one dimensional and the space as three dimensional. Both together they form the four-dimensional pseudo-Euclidean non-compact continuum $M_4 \subset M_6$

$$M_4 = T_1 \oplus E_3 \tag{2.2}$$

with the metric $(+---)$. The symbol T_1 stands for the one-dimensional continuum of events along a chosen direction in the continuum T_3.

The physics on M_4, when only bradyons are considered, is just the usual relativity. It will turn out that when tachyons are taken into account, the real Minkowski space M_4 is not enough for a complete description. If we wish to avoid complex physical quantities, we must introduce the symmetric space–time. Moreover, the three dimensionality of time becomes an observable reality.

3. Subluminal and superluminal transformations

When a reference frame S is specified, an event $\mathscr{P}$ in M_6 is described by six coordinates

$$x^a \equiv (x^{\bar{r}}, x^r) \equiv (t^r, x^r) \equiv (\boldsymbol{t}, \boldsymbol{x}) \tag{3.1}$$

where $a = \bar{1}, \bar{2}, \bar{3}, 1, 2, 3$; $\bar{r} = \bar{1}, \bar{2}, \bar{3}$; $r = 1, 2, 3$; $x^{\bar{r}} \equiv t^r$.

The square of the distance between two infinitesimally separated events with the coordinates x^a and $x^a + \mathrm{d}x^a$ is given by the quadratic form

$$\mathrm{d}s^2 \equiv g_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b = \mathrm{d}x^a\, \mathrm{d}x_a = \mathrm{d}x^{\bar{r}}\, \mathrm{d}x_{\bar{r}} + \mathrm{d}x^r\, \mathrm{d}x_r. \tag{3.2}$$

In an inertial frame the metric tensor g_{ab} is

$$\delta_{ab} = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} \tag{3.3}$$

where $\underline{1}$ is the three-dimensional unit matrix

$$\underline{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

Then the quadratic form (3.2) can be written as

$$\mathrm{d}s^2 = \mathrm{d}x^{\bar{r}}\, \mathrm{d}x^{\bar{r}} - \mathrm{d}x^r\, \mathrm{d}x^r \equiv \mathrm{d}t^2 - \mathrm{d}x^2 \qquad c = 1 \tag{3.5}$$

where $\mathrm{d}t^2 \equiv |\mathrm{d}\boldsymbol{t}|^2 \equiv \mathrm{d}t^r\, \mathrm{d}t_r = \mathrm{d}t^r\, \mathrm{d}t^r$ and $\mathrm{d}x^2 \equiv |\mathrm{d}\boldsymbol{x}|^2 = -\mathrm{d}x^r\, \mathrm{d}x_r = \mathrm{d}x^r\, \mathrm{d}x^r$. (3.6)

We distinguish three types of the quadratic forms (3.5):

(i) $\mathrm{d}s^2 = 0$ null distance

(ii) $\mathrm{d}s^2 > 0$ time-like distance

(iii) $\mathrm{d}s^2 < 0$ space-like distance.

As in four-dimensional relativity, case (i) is also satisfied in six-dimensional relativity by the events connected by *the light signals*. Let the speed of light be defined as

$$c = (\mathrm{d}x^r\,\mathrm{d}x^r)^{1/2}/(\mathrm{d}t^r\,\mathrm{d}t^r)^{1/2} \equiv \mathrm{d}x/\mathrm{d}t.$$

We use units in which $c = 1$.

Case (ii) is assumed to be satisfied by the events along the world line of a *bradyon*.

Case (iii) is assumed to be satisfied by the events along the world line of a *tachyon*.

A transformation that preserves the sign of $\mathrm{d}s^2$ has been called a *subluminal transformation* (Parker 1969, Recami and Mignani 1974). An object O, which appears as a bradyon $B = O(S)$ in a frame S, appears as a bradyon $B' = O(S')$ in another frame S', associated to S by a subluminal transformation. A transformation that changes the sign of $\mathrm{d}s^2$ has been called a *superluminal* transformation. An object O, which appears as a bradyon $B = O(S)$ in a frame S, appears as a tachyon $T = O(S^*)$ in a frame S^*, associated to S by a superluminal transformation.

4. Some explicit expressions of the transformations

A general transformation which preserves $\mathrm{d}s^2$ in equation (3.5), apart from the sign, has been discussed by Cole (1980). In general, a homogeneous transformation in six dimensions has 15 parameters. The motion of a particle may be specified by its six-velocity $u^a \equiv \mathrm{d}x^a/\mathrm{d}s = v^a/(v^b v_b)^{1/2}$, with $v^a \equiv (m^r, v^r) \equiv (\boldsymbol{\alpha}, \boldsymbol{v})$, where $m^r \equiv \boldsymbol{\alpha} = \mathrm{d}\boldsymbol{t}/\mathrm{d}t$ is a unit vector in three-dimensional time, and $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is the velocity in three-dimensional space. Obviously $v^a v_a = m^r m_r + v^r v_r = 1 - v^2 = (\mathrm{d}s/\mathrm{d}t)^2$. Let S and S' be two frames with spatial origins given by the world lines O and O', respectively. Let, according to Cole, the motion of O be specified by $(\boldsymbol{\alpha}_0, \boldsymbol{0})$ and $(\boldsymbol{\alpha}'_0, \boldsymbol{v}')$ in S and S', respectively, and let the motion of O' be specified by $(\boldsymbol{\alpha}_{0'}, \boldsymbol{v})$ and $(\boldsymbol{\alpha}'_{0'}, \boldsymbol{0})$ in S and S' respectively. If we exclude the spatial and time rotations, then there remain nine independent parameters, characterising a Lorentz transformation in M_6. These parameters are given by $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}'_0, \boldsymbol{\alpha}_{0'}, \boldsymbol{\alpha}'_{0'}, \boldsymbol{v}$ and $\boldsymbol{v}'$, which quantities are not all independent, but related according to equations (2.4) and (2.5) of Cole's (1980) paper. One relation can be immediately found by observing that the scalar product of the six-velocities u_0^a and $u_{0'}^a$ is invariant, apart from the sign ($k = 1$ for subluminal, $k = -1$ for superluminal transformations)

$$u_0^a u_{0'a} = k u_0'^a u'_{0'a} \tag{4.1}$$

i.e.

$$(\boldsymbol{\alpha}_{0'}\boldsymbol{\alpha}_0 - \boldsymbol{v}_{0'}\boldsymbol{v}_0)/[(1 - v_{0'}^2)(1 - v_0^2)]^{1/2} = k(\boldsymbol{\alpha}'_{0'}\boldsymbol{\alpha}'_0 - \boldsymbol{v}'_{0'}\boldsymbol{v}'_0)/[(1 - v_{0'}'^2)(1 - v_0'^2)]^{1/2}$$

which reduces to

$$\boldsymbol{\alpha}_{0'}\boldsymbol{\alpha}_0\gamma = k\boldsymbol{\alpha}'_{0'}\boldsymbol{\alpha}'_0\gamma' \qquad \text{for } \boldsymbol{v}_0 = \boldsymbol{v}'_{0'} = 0 \tag{4.2}$$

where $\gamma = (1 - v^2)^{-1/2} = \mathrm{d}t_{0'}/\mathrm{d}t'_{0'}$ and $\gamma' = (1 - v'^2)^{-1/2} = \mathrm{d}t'_0/\mathrm{d}t_0$ with $\boldsymbol{v} \equiv \boldsymbol{v}_{0'}$ and $\boldsymbol{v}' \equiv \boldsymbol{v}'_0$.

Let us assume that

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}'_0 \qquad \text{and} \qquad \boldsymbol{\alpha}_{0'} = \boldsymbol{\alpha}'_{0'} \tag{4.3}$$

which reduces the number of independent parameters from nine to five. This assumption has its analogy in the usual four-dimensional Lorentz transformations where we assume that in the new frame *the direction* of velocity of O' is the same as in the old frame (i.e. we exclude rotations of space axes). From equation (4.3) it follows that in the subluminal case $\gamma = \gamma'$, that is $|\boldsymbol{v}| = |\boldsymbol{v}'|$. The same result can be obtained also by assuming

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_{0'} \qquad \text{and} \qquad \boldsymbol{\alpha}'_0 = \boldsymbol{\alpha}'_{0'}. \tag{4.4}$$

The first assumption (4.3) concerns only the way we orient the new axes $\boldsymbol{t}'$, without affecting the physical content, that is the orientation of the world line O'. On the other hand, the assumption (4.4) restricts the world lines O and O' to lie in a certain prescribed hypersurface M_4 defined by the axes $(\boldsymbol{\alpha}_0, \boldsymbol{x})$.

Let $\boldsymbol{m} \equiv m^r \equiv \boldsymbol{\alpha}_0$ be a time unit 3-vector of O, and let $\boldsymbol{n} \equiv n^r$ be a space unit 3-vector in the direction of $\boldsymbol{v}$, satisfying $m^r m_r = 1$ and $n^r n_r = -1$, respectively. A transformation $L^a{}_b$ which satisfies the condition (4.3) can be obtained by:

(i) performing the spatial, $R(\boldsymbol{n})$, and the temporal, $R(\boldsymbol{m})$, rotations which turn the axes t^r, x^r so that the new axis t''^1 lies along $\boldsymbol{m}$, and x''^1 along $\boldsymbol{n}$;

(ii) performing the subluminal boost in the (t''^1, x''^1) plane;

(iii) performing the spatial and temporal rotations which are inverse to the rotations (i).

The result is the *subluminal transformation*

$$L^a{}_b(v) = \begin{pmatrix} \delta^{\bar r}{}_{\bar s} + Av^2 m^{\bar r} m_{\bar s} & \gamma_v v m^{\bar r} n_s \\ -\gamma_v v n^r m_{\bar s} & \delta^r{}_s - Av^2 n^r n_s \end{pmatrix} \tag{4.5}$$

where $\gamma_v \equiv \gamma$ and $A = \gamma^2/(1+\gamma)$. Here $v = v_A < 1$ is the velocity of a world line A, with $\boldsymbol{\alpha}_A = \boldsymbol{\alpha}_0$, lying in the hyperplane $(\boldsymbol{\alpha}_0, \boldsymbol{x})$. From condition (4.3), this velocity v_A is the same as the velocity v of O' which does not necessarily lie in this hyperplane, that is $\boldsymbol{\alpha}_{0'} \neq \boldsymbol{\alpha}_0$. This can be proved from the obvious relation $\gamma_A = \mathrm{d}t_A/\mathrm{d}t'_A = \mathrm{d}t_{0'}\boldsymbol{\alpha}_{0'}\boldsymbol{\alpha}_0/(\mathrm{d}t'_{0'}\boldsymbol{\alpha}'_{0'}\boldsymbol{\alpha}'_0) = \gamma\boldsymbol{\alpha}_{0'}\boldsymbol{\alpha}_0/(\boldsymbol{\alpha}'_{0'}\boldsymbol{\alpha}'_0)$ by taking into account equation (4.3). The transformation (4.5), therefore, does not hold only in the special case of $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_{0'}$, but also in the more general case of $\boldsymbol{\alpha}_0 \neq \boldsymbol{\alpha}_{0'}$, provided the condition (4.3) is satisfied. The inverse transformation has the same form (4.5) with the replacement $v \to -v$. The matrix (4.5) satisfies the orthogonality condition

$$L^a{}_b(v) L_a{}^c(v) = \delta_b{}^c \tag{4.6}$$

and therefore preserves $\mathrm{d}s'^2 = \mathrm{d}s^2$.

It is not difficult now to find the matrix $\tilde{L}^a{}_b(w)$, with $w < 1$, which satisfies

$$\tilde{L}^a{}_b(w) \tilde{L}_a{}^c(w) = -\delta_b{}^c. \tag{4.7}$$

It is given by

$$\tilde{L}^a{}_b(w) = \begin{pmatrix} -\gamma_w w n^r m_s & \delta^r{}_s - Bw^2 n^r n_s \\ \delta^r{}_s + Bw^2 m^r m_s & \gamma_w w m^r n_s \end{pmatrix} \tag{4.8}$$

where $\gamma_w = (1-w^2)^{-1/2}$ and $B = \gamma_w^2/(1+\gamma_w)$. One can verify directly from equation (4.8) that $\tilde{L}^a{}_b(w)$ changes the sign of the quadratic form, and is therefore a *superluminal* *transformation.* The matrix (4.8) can be obtained from (4.5) by interchanging $t'^r \to x'^r$ and $x'^r \to t'^r$ in the relation $dx'^a = L^a{}_b(\nu)\, dx^b$, so that

$$\nu = \left.\frac{dx}{dt}\right|_{dx'=0} = \left.\frac{dt}{dx}\right|_{dt'=0} = v \to \nu = \left.\frac{dx}{dt}\right|_{dt'=0} = \left.\frac{dt}{dx}\right|_{dx'=0} = \frac{1}{v} = w \tag{4.9}$$

where the subscripts refer to the world lines along which in S' $dx' = 0$ and $dt' = 0$, respectively.

Alternatively, the transformation which changes $ds'^2 = -ds^2$ can be given by the complex matrix

$$L^{*a}{}_b(v) = -iL^a{}_b(v) \qquad v > 1. \tag{4.10}$$

If we orient the coordinate system so that $\boldsymbol{m} = (1, 0, 0)$, $\boldsymbol{n} = (1, 0, 0)$, and if the spatial origins of S and S' coincide at $t^r = 0$ so that $x_0^a = x_{0'}^a = 0$, we obtain from equation (4.5) *the subluminal boost in six dimensions* (Cole 1977)

$$\begin{aligned} t'^1 &= \gamma_v(t^1 - vx^1) \qquad & t'^2 &= t^2 \qquad & t'^3 &= t^3 \\ x'^1 &= \gamma_v(x - vt^1) \qquad & x'^2 &= x^2 \qquad & x'^3 &= x^3 \end{aligned} \tag{4.5a}$$

from equation (4.8) the *real superluminal boost* (Cole 1977)

$$\begin{aligned} t'^1 &= \gamma_w(x^1 - wt^1) = (vx^1 - t^1)/(v^2-1)^{1/2} \qquad & t'^2 &= x^2 \qquad & t'^3 &= x^3 \\ x'^1 &= \gamma_w(t^1 - wx^1) = (vt^1 - x^1)/(v^2-1)^{1/2} \qquad & x'^2 &= t^2 \qquad & x'^3 &= t^3 \end{aligned} \tag{4.8a}$$

and from equation (4.10) the *complex superluminal boost*

$$\begin{aligned} t'^1 &= -i\gamma_v(t^1 - vx^1) = (vx^1 - t^1)/(v^2-1)^{1/2} \qquad & t'^2 &= -it^2 \qquad & t'^3 &= -it^3 \\ x'^1 &= -i\gamma_v(x^1 - vt^1) = (vt^1 - x^1)/(v^2-1)^{1/2} \qquad & x'^2 &= -ix^2 \qquad & x'^3 &= -ix^3. \end{aligned} \tag{4.10a}$$

5. Contraction of the six-dimensional equations into the observable four-dimensional equations

Now we must take into account the fact that our instruments and our brain, when only bradyons are present, do not register the three dimensionality of time, but only of space. In other words, from the six-dimensional equations we must obtain the usual four-dimensional subluminal equations. As pointed out by Cole (1980) this is achieved by imposing parallel time directions in each frame. I shall now elaborate this topic more explicitly.

Let an event $\mathscr{P}$ be projected into the subspace $T_3 \subset M_6$. We obtain the point $\mathscr{T} \in T_3$. The distance $\overline{O\mathscr{T}}$ between the coordinate origin O and the point $\mathscr{T}$ is equal to $|\boldsymbol{t}| = (t^r t_r)^{1/2}$. The differential of $|\boldsymbol{t}|$ is

$$d|\boldsymbol{t}| = t^r\, dt_r/|\boldsymbol{t}| = \boldsymbol{t}\, d\boldsymbol{t}/|\boldsymbol{t}| = |\boldsymbol{t}|\, |d\boldsymbol{t}| \cos\varphi/|\boldsymbol{t}| = |d\boldsymbol{t}| \cos\varphi \tag{5.1}$$

where φ is the angle between the vectors $\boldsymbol{t}$ and $d\boldsymbol{t}$.

We shall assume that an observer and his instruments are such that he observes $|d\boldsymbol{t}| \equiv dt$, but not the components dt_1, dt_2, dt_3 separately. The observable time is therefore $\int |d\boldsymbol{t}|$. Some authors (Mignani and Recami 1976) have assumed that the observable time is $|\boldsymbol{t}| = (t_1^2 + t_2^2 + t_3^2)^{1/2}$ but then some others have taken for the observable time interval $d|\boldsymbol{t}|$ of equation (5.1) instead of $|d\boldsymbol{t}|$ of equation (3.6). Such an assumption is true only for 'radial' displacement $d\boldsymbol{t}$ (when $\cos\varphi = 1$), and when taken as being general it leads to the non-invariance of the speed of light or to nonlinearity of the transformations, and several other troubles discussed in the literature, all of which we avoid by the assumption that the observable time interval is given by (3.6).

In the case of a generic 6-vector p^a which transforms as

$$p'^a = L^a{}_b p^b \tag{5.2}$$

the observed quantities are

$$\begin{aligned} &p^\mu \equiv (p^0, p^r) &&\mu = 0, 1, 2, 3 \\ &p^0 \equiv (p^{\bar r} p_{\bar r})^{1/2} &&\bar r = \bar 1, \bar 2, \bar 3 \qquad r = 1, 2, 3 \end{aligned} \tag{5.3}$$

where p^μ must transform as the usual 4-vector. The requirements are

$$p^{\bar s} p_{\bar s} = (p^0)^2 \qquad p'^{\bar s} p'_{\bar s} = (p'^0)^2 \tag{5.4}$$

$$p'^\mu = L^\mu{}_\nu p^\nu. \tag{5.5}$$

Bearing in mind equation (5.2) we see that, in order to satisfy the requirements (5.4), (5.5), the 6-momentum (say!) p^a and the transformation matrix $L^a{}_b$ must satisfy the conditions

$$L^r{}_0 p^0 = L^r{}_{\bar s} p^{\bar s} \tag{5.6a}$$

$$L^{\bar s}{}_{\bar r} L_{\bar s}{}^{\bar m} p^{\bar r} p_{\bar m} = (L^0{}_0 p^0)^2 \tag{5.6b}$$

$$L^{\bar s}{}_m L_{\bar s}{}^n = L^0{}_m L_0{}^n \tag{5.6c}$$

$$L^{\bar s}{}_r L_{\bar s}{}^{\bar m} p^r p_m = L^0{}_r p^r L_0{}^0 p^0. \tag{5.6d}$$

The conditions (5.6) imply also the fulfilment of the orthogonality relation

$$L^\mu{}_\alpha L_\mu{}^\beta = \delta_\alpha{}^\beta. \tag{5.7}$$

By direct calculation we can find out that a generic 6-vector p^a and the transformation $L^a{}_b$ given by equation (4.5) satisfies the conditions (5.6*a*), (5.6*b*) and (5.6*d*), provided that

$$m^{\bar r} p_{\bar r} = p^0 \tag{5.8}$$

i.e.

$$m^{\bar r} p_{\bar r} = m^0 p^0 \cos\beta \qquad \cos\beta = 1.$$

The condition (5.6*c*) is identically fulfilled, regardless of p^a.

The relation (5.8) means that m^r and p^r are parallel. As already stated, the transformation $R(\boldsymbol{m})$ rotates $t_1 \to t''_1$, so that the $t''_{\bar 1}$ axis points in the direction of $m^{\bar r}$. According to (5.8), $p^{\bar r}$ must also have the same direction in T_3. The Lorentz boost is then performed with respect to the direction of $t''_{\bar 1}$. Instead of considering the whole six-dimensional space–time, it is enough, due to equation (5.8), to consider only a four-dimensional Minkowski subspace M_4. The latter is the direct sum of three-dimensional Euclidean space E_3 and one-dimensional time, defined by the direction $m^{\bar r}$ which is unique for all 6-vectors observable to a given observer. In other words, equation (5.8) implies that all 6-vectors p^a, which are physically accessible (i.e. measurable) to a given observer, are situated on a four-dimensional Minkowski sheet embedded in M_6. All other possible 6-vectors, not lying on M_4, are not directly accessible to a subluminal observer on M_4, nevertheless, they are detectable by a superluminal observer whose Minkowski space–time M'_4 is 'orthogonal' to M_4, as will soon be clear.

First, let us apply the condition (5.8) to $p'^a = L^a{}_b(v)p^b$, where $L^a{}_b(v)$ is given by (4.5) and where $p^a \equiv (p^{\bar{r}}, p^r) = (E^r, p^r)$ is a generic 6-vector, for instance energy–momentum. We obtain the 'contracted' subluminal transformations

$$E' = \gamma_v(E + vn_sp^s) \qquad E \equiv p^0$$
$$p'^r = -\gamma_v n^r E + p^r - Av^2 n^r n_s p^s \qquad vn_s = v_s = -v^s. \tag{5.9}$$

This is exactly the well known general transformation in M_4.

If we apply $m^{\bar{r}}p_{\bar{r}} = p^0$ and $m'^{\bar{r}}p'_{\bar{r}} = p'^0$ to $p'^a = \tilde{L}^a{}_b(w)p^b$, where $\tilde{L}^a{}_b(w)$ is given by (4.8), we obtain the contracted superluminal transformations as considered respectively from

(i) the viewpoint of a subluminal observer S

$$p' = \gamma_w(E + wn_sp^s)$$
$$E'^r = p^r - \gamma_w wn^r E - Bw^2 n^r n_s p^s \tag{5.10}$$

(ii) the viewpoint of a superluminal observer S'

$$p'^r = E^r + Bw^2 m^r m_s E^s - \gamma_w wpm^r$$
$$E' = \gamma_w(p - wE^s m_s). \tag{5.11}$$

Analogously, we can also contract the complex superluminal transformations (4.10) by imposing $m^rE_r = E$. We obtain

$$E' = -\mathrm{i}\gamma_v(E + vp^s n_s)$$
$$p'^r = -\mathrm{i}(-\gamma_v vn^r E + p^r - Av^2 n^r n_s p^s). \tag{5.12}$$

If $n^r = (1, 0, 0)$ these last equations (5.12) are identical to the Recami–Mignani (1974) superluminal transformations in four dimensions.

By assuming that a subluminal observer S can observe only those time vectors $p^{\bar{r}}$ which are parallel to a certain time direction $m^{\bar{r}}$, and that a superluminal observer S' can observe only those time vectors $p'^{\bar{r}}$ which are parallel to a certain time direction $m'^{\bar{r}}$ in S', it follows that the observable events of S are not the same as the observable events of S'. In special relativity there are two classes of observable events: $\{P\}$ on M_4 observed by a subluminal observer S and $\{P'\}$ on M'_4 observed by a superluminal observer S'.

This result can also be expressed by saying that there is no such four-dimensional subspace $M_4 \subset M_6$ and no *real* linear four-dimensional transformation which induces the change of a 4-vector sign $x^\mu x_\mu \to -x^\mu x_\mu$. Only complex transformations do this job. This is the physical background of the imaginary unit i entering the superluminal transformations in M_4. Other workers have closely approached this interpretation, namely that the factor i makes some quantities unobservable (Corbèn 1976). The papers by Recami and Maccarrone (1980) and by Caldirola *et al* (1980) in which the authors essentially obtain the result that a superluminal transformation transforms a sphere into a hyperbola can be understood in the following way: a cross section of a suitable six-dimensional object (i) with M_4 is a sphere, (ii) with M'_4 is a hyperboloid. This difference in cross sections results from the fact that (*a*) M_4 and M'_4 are *orthogonal* (in the six-space M_6) to each other, and (*b*) M_6 is non-compact. Similarly two suitable orthogonal cross sections of a cone by plane give a circle and a hyperbola, respectively.

Occurrence of imaginary units when passing from a subluminal to a superluminal frame in M_4 is equivalent to the assertion that in non-compact M_6 superluminal (i.e. the quadratic form sign changing) transformations are real but the observable spaces M_4 and M'_4 are orthogonal to each other.

6. General linear non-orthogonal covariance transformations

I have always been fascinated with the idea (Pavšič 1979a, b, 1980, Pavšič and Recami 1977) why to consider as covariance transformations only those orthogonal transformations which preserve the length of a vector. Why not consider (in special relativity) all possible linear transformations, at least as a theoretical possibility to be explored? A superluminal transformation which changes $\mathrm{d}s^2 \to -\mathrm{d}s^2$ is an example of such a more general transformation.

Let us consider the transformation

$$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu} \tag{6.1}$$

where the constant elements $a^{\mu}{}_{\nu}$ are not restricted by any constraint. For the moment I do not specify the metric or the dimension of space. Let an object O be specified by the set of events $\{E_i\}$ $(i = 1, 2, \ldots, N)$. The transformation (6.1) transforms

(i) in the passive sense

$$O(S) \to O(S') \qquad \{E_i(S)\} \to \{E_i(S')\} \qquad \{x_i^{\mu}\} \to \{x'_i = a^{\mu}{}_{\nu} x_i{}^{\nu}\}$$

(ii) and in the active sense

$$O(S) \to O'(S) \qquad \{E_i(S)\} \to \{E'_i(S)\} \qquad \{x_i^{\mu}\} \to \{x_i'^{\mu} = a^{-1\mu}{}_{\nu} x_i^{\nu}\}.$$

The notation $O(S)$ means that O is observed from the frame S, etc. The transformed object O' is *equivalent* to O; the set of events $\{E_i\}$ is equivalent to the transformed set $\{E'_i\}$. The objects $\{O, O', O'', \ldots\}$ form an *equivalence class* of objects which all transform into each other by a covariance transformation. For further details see Pavšič (1980).

Two equivalent objects O_1 and O_2 can be specified, in a given frame S, by two different *state matrices* $b^{\mu}_{1\nu}$ and $b^{\mu}_{2\nu}$, specifying their respective states within the equivalence class. For instance, if the objects have different orientations, then $b^{\mu}_{1\nu}$ and $b^{\mu}_{2\nu}$ specify their orientation states; if they have different velocities, then the state matrices specify their velocity states, i.e. their 'orientations' in space–time, etc.

Though the coordinates x^{μ} change according to (6.1), we can introduce *the proper coordinates*

$$\xi^{\mu} = b^{\mu}{}_{\nu} x^{\nu} \tag{6.2}$$

which are invariant under any covariance transformation (6.1):

$$\xi'^{\mu} = \xi^{\mu}. \tag{6.3}$$

Then from (6.1), (6.2) and (6.3) it follows that

$$b'^{\mu}{}_{\alpha} = a^{-1\nu}{}_{\alpha} b^{\mu}{}_{\nu}. \tag{6.4}$$

Further examinations will eventually show whether this extended group has any relation (Pavšič 1979b, 1980) with some presently observed (or not yet observed) objects and internal symmetries like colour, flavour, etc, which at the moment are treated only at the formal level with gauge theories, without any deeper (e.g. space–time) understanding of them. In the present paper I shall restrict myself only to the case of the dilatation and apply it to bradyons and tachyons.

7. A special case: the imaginary dilatation and its relation to the superluminal transformations

The action of the dilatation on the coordinates x^a is defined by

$$x'^a = D^a{}_b x^b = \rho x^a \tag{7.1}$$

where we have represented the dilatation by

$$D^a{}_b = \begin{pmatrix} \underline{\rho} & \underline{0} \\ \underline{0} & \underline{\rho} \end{pmatrix} \qquad \underline{\rho} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}. \tag{7.2}$$

Two equivalent objects O_1 and O_2, which can be mapped into each other (Pavšič 1980) by a dilatation $D^a{}_b$, can be specified, in a given frame S, by two different dilatational state matrices or simply scales C^a_{1b} and C^a_{2b}, respectively. These are special cases of the state matrix of the previous section. Instead of the proper coordinates ξ^a, which are invariant under any covariance transformation, it is now more convenient to define the coordinates

$$\eta^a = C^a{}_b x^b = \kappa x^a \tag{7.3}$$

which are invariant only under dilatations (7.1):

$$\eta'^a = \eta^a. \tag{7.4}$$

In equation (7.3), the scale is represented by the matrix

$$C^a{}_b = \begin{pmatrix} \underline{\kappa} & \underline{0} \\ \underline{0} & \underline{\kappa} \end{pmatrix} \qquad \underline{\kappa} = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix}. \tag{7.5}$$

From (7.1), (7.3) and (7.4) it then follows that

$$C'^a{}_b = D^{-1c}{}_b C^a{}_c \qquad \text{or} \qquad \kappa' = \rho^{-1}\kappa. \tag{7.6}$$

Let the quadratic form in the coordinates η^a be

$$\mathrm{d}\sigma^2 \equiv \delta_{ab}\, \mathrm{d}\eta^a\, \mathrm{d}\eta^b = \mathrm{d}\eta^a\, \mathrm{d}\eta_a \tag{7.7}$$

where δ_{ab} is given by equation (3.3). Let $\mathscr{L}^a{}_b$ be a homogeneous transformation, with $\det \mathscr{L}^a{}_b = 1$, which preserves the quadratic form (7.7)

$$\eta'^a = \mathscr{L}^a{}_b \eta^b \tag{7.8}$$

and consequently satisfies

$$\mathscr{L}^a{}_b \mathscr{L}_a{}^c = \delta_b{}^c. \tag{7.9}$$

The invariance of the quadratic form (7.7) reads

$$\mathrm{d}\sigma'^2 = \mathrm{d}\sigma^2. \tag{7.10}$$

If κ is constant, then from (7.3) and (7.10) it follows that

$$\begin{aligned} \mathrm{d}\sigma'^2 &= C'^a{}_b C'_a{}^c \, \mathrm{d}x'^b \, \mathrm{d}x'_c = \kappa'^2 \, \mathrm{d}x'^a \, \mathrm{d}x'_a \\ &= \kappa^2 \, \mathrm{d}x^a \, \mathrm{d}x_a = C^a{}_b C_a{}^c \, \mathrm{d}x^b \, \mathrm{d}x_c = \mathrm{d}\sigma^2. \end{aligned} \tag{7.11}$$

Bearing in mind (7.1) and (7.6) we see that equation (7.11) is equivalent to

$$\mathrm{d}x'^a \, \mathrm{d}x'_a = D^a{}_b D_a{}^c \, \mathrm{d}x^b \, \mathrm{d}x_c = \rho^2 \, \mathrm{d}x^a \, \mathrm{d}x_a. \tag{7.12}$$

In the present paper we are interested in two special cases

$$D^a{}_c D_a{}^c = \begin{cases} \delta_b{}^c \\ -\delta_b{}^c \end{cases} \tag{7.13a, b}$$

where $\delta_b{}^c$ is the unit 6×6 matrix. The case (7.13a) implies $\mathrm{d}s'^2 = \mathrm{d}s^2$. The transformation $\mathscr{L}^a{}_b$ is then a *subluminal transformation.* The case (7.13b) implies $\mathrm{d}s'^2 = -\mathrm{d}s^2$; $\mathscr{L}^a{}_b$ is then a *superluminal transformation.*

Let us find particular representations of $D^a{}_b$ for the cases (7.13a) and (7.13b). In the *subluminal case* $D^a{}_b$ can be represented by the unit 6×6 matrix $\begin{pmatrix} 1 & 0 \\ \underline{0} & \underline{1} \end{pmatrix}$. In the *superluminal case* $D^a{}_b$ can be represented by the *diagonal matrix* $D^a{}_b = \begin{pmatrix} -\mathrm{i} & 0 \\ \underline{0} & \underline{-\mathrm{i}} \end{pmatrix}$, $D^a{}_b = D_a{}^b$. Equivalently, it can be represented by the *non-diagonal real matrix* $D^a{}_b = \begin{pmatrix} 0 & 1 \\ 1 & \underline{0} \end{pmatrix}$, $D_a{}^b = -D^a{}_b$. Let us choose $C^a{}_b = \begin{pmatrix} 1 & 0 \\ \underline{0} & \underline{1} \end{pmatrix}$ for a *bradyon.* Then, according to (7.6), $C^a{}_b = \begin{pmatrix} \mathrm{i} & 0 \\ \underline{0} & \mathrm{i} \end{pmatrix}$ for a *tachyon* in the diagonal representation and $C^a{}_b = \begin{pmatrix} 0 & 1 \\ \underline{1} & \underline{0} \end{pmatrix}$ in the non-diagonal representation.

The relation (7.8) can be written in terms of the usual coordinates

$$x'^a = L^a{}_b x^b \tag{7.8a}$$

where using (7.3) we obtain

$$L^a{}_b = C'^{-1a}{}_c C^d{}_b \mathscr{L}^c{}_d. \tag{7.14}$$

From (7.6), (7.9) and (7.14) it follows that

$$L^a{}_b L_a{}^c = D^a{}_b D_a{}^c \tag{7.15}$$

which embraces both the subluminal and the superluminal case, depending on the type of $D^a{}_b$ given in equations (7.13a) and (7.13b).

From the above considerations it follows that *both subluminal and superluminal transformations can be written by the same matrix when expressed in the dilatation invariant coordinates*

$$\mathscr{L}^a{}_b = \begin{pmatrix} \delta^{\bar r}{}_{\bar s} + A v^2 m^{\bar r} m_{\bar s} & \gamma_v v m^{\bar r} n_s \\ -\gamma_v v n^r m_{\bar s} & \delta^r{}_s - A v^2 n^r n_s \end{pmatrix} \tag{7.16}$$

where $A = \gamma_v^2/(1+\gamma_v)$ and $\gamma_v = (1-v^2)^{-1/2}$. It has $\det \mathscr{L}^a{}_b = 1$ and satisfies (7.9). The parameters of the transformation are the time unit vector $m^{\bar r}$, the space unit vector n^r and

$$v = \mathrm{d}\eta/\mathrm{d}\tau\big|_{\mathrm{d}\eta'=0} = \mathrm{d}\tau/\mathrm{d}\eta\big|_{\mathrm{d}\tau'=0} \tag{7.17}$$

where $\mathrm{d}\eta \equiv (\mathrm{d}\eta^r \, \mathrm{d}\eta^r)^{1/2}$, $\mathrm{d}\tau \equiv (\mathrm{d}\tau^r \, \mathrm{d}\tau_r)^{1/2}$. Equation (7.17) can be obtained from $\mathrm{d}\eta'^a = \mathscr{L}^a{}_b \, \mathrm{d}\eta^b$ by imposing $\mathrm{d}\eta' = 0$ and $\mathrm{d}\tau' = 0$, respectively.

Let us choose $C^a{}_b = \begin{pmatrix} 1 & 0 \\ \underline{0} & \underline{1} \end{pmatrix}$. If we insert into equation (7.14)

(i) $C'^a{}_b = \begin{pmatrix} 1 & 0 \\ \underline{0} & \underline{1} \end{pmatrix}$, then $L^a{}_b$ of equations (7.8a) and (7.14) becomes the *subluminal transformation* $L^a{}_b(v)$ of equation (4.5),

(ii) $C'^a{}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $L^a{}_b$ becomes the *real superluminal transformation* $\tilde{L}^a{}_b(w)$ of equation (4.8) with $w = 1/v = \nu$, consistent with the definition of ν in equation (7.17),

(iii) $C'^a{}_b = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, then $L^a{}_b$ becomes the *complex superluminal transformation* $L^{*a}{}_b(v)$ of equation (4.10).

Instead of treating bradyons and tachyons and the transformations between them separately, we can always use relation (7.8) and the transformation matrix (7.16), bearing in mind suitable scales $C^a{}_b$ and $C'^a{}_b$. When we wish to transform η^a and $\mathscr{L}^a{}_b$ back to x^a and $L^a{}_b$, we must apply equations (7.3) and (7.14). So we can reproduce all three types of transformations (4.5), (4.8) and (4.10), the special cases of which are (4.5*a*), (4.8*a*) and (4.10*a*).

8. Conclusion

In the present paper I have written an explicit homogeneous transformation matrix which satisfies the condition that the world line $O(O')$ representing the spatial origin of a frame $S(S')$ has the same time direction in both S and S'; the time directions of O and O', as measured in one frame, are not necessarily the same. Since an observer perceives only one-dimensional time, the six-dimensional equations have been contracted into four-dimensional equations. All equations of the usual four-dimensional relativity can be recovered, if we assume that only those 6-vectors are observable which in three-dimensional time are parallel to a certain time direction $\boldsymbol{m}$ defining a Minkowski space M_4. All other 6-vectors are not observable for the class of (subluminal) observers on M_4. By assuming the complete symmetry between subluminal and superluminal observers and the corresponding laws, we came to the result that the Minkowski space M'_4 of a superluminal observer S' is 'orthogonal' to the Minkowski space M_4 of a corresponding subluminal observer S. Therefore, it is meaningless to speak about real superluminal transformations in M_4, and this reflects itself in the fact that these transformations are necessarily complex. Those coordinates (e.g. t^1 and x^1) which remain real span the cross section $M_3 \cap M'_4$ which is common both for S and S', whilst other coordinates which become imaginary under the transformation describe the events which are not commonly observed by S and S'.

Next, I have established a unified formalism both for subluminal and superluminal transformations. The essence is the observation that an extended bradyon at rest and an infinite-speed tachyon are equivalent (six-dimensional) objects, distinguished by the orientations of their space-like and time-like axes. This orientation is specified by the quantity $C^a{}_b$ which has been called scale. By ascribing to bradyons and tachyons suitable scales both types of objects and transformations can be described by the same equations in terms of the dilatation invariant coordinates η^a. When we go back to the usual coordinates x^a, this description splits into two separate cases, a subluminal and a superluminal one. In six dimensions the dilatation $D^a{}_b$, a special case of superluminal transformations, can be represented either by a diagonal imaginary matrix or a non-diagonal real matrix. In fact, the matrix $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ can be transformed into the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by a unitary transformation. Both representations of the dilatation and the tachyonic scale, the diagonal and the non-diagonal one, give identical kinematical results. In future it will also be necessary to explore the dynamics, especially the interactions between bradyons and tachyons. It is reasonable to hope that the unified formalism presented here will greatly facilitate this task.

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